Sets and relations

1. Let $A = \{1, 2, 5, 6, 7, 10, 11, 12\}$, $B = \{x \mid x \text{ is an odd integer and } 3 \leq x \leq 10\}$, and $C = \{a, b, c, 4, 5, 6\}$.

(a) Fill in the blanks $\_$ with the most suitable symbol ($\in, \notin, \subseteq, \nsubseteq$):
$7 \_ A$, $6 \_ B$, $10 \_ B$, $4 \_ A$, $b \_ C$, $b \_ B$, $\{2, 5, 12\} \_ A$, $\{3, 5, 6\} \_ B$, $\{a, b, 5\} \_ C$.

(b) Compute $A \cup B$, $A \setminus B$, $\mathcal{P}(A \cap B)$, $(A \cap B) \times \{a, b\}$.

2. (a) Let A and B be given sets. Show that $A \cap (B \setminus A) = \emptyset$. Explain each step of your proof.

(b) Let A and B be subsets of the universal set U. Show that $A \subseteq B$ if and only if $\overline{B} \subseteq \overline{A}$. Explain each step of your proof.

3. Justify whether the statement

$$(A \cup B) \cap (A \cup C) \subseteq \overline{A} \cup C$$

holds for arbitrary sets A, B, and C. Explain each step of your proof.

4. Let A be a non-empty set. Determine which of the sets

$$\emptyset, \{\emptyset\}, A, \{A\}, \{A, \emptyset\}$$

are elements and which are subsets of the sets $\mathcal{P}(A)$ and $\mathcal{P}(\mathcal{P}(A))$.

5. Justify whether the following statement is true or false:

If $\mathcal{P}(X) = \mathcal{P}(Y)$ for sets X and Y, then X and Y are equal.

6. In the power set $\mathcal{P}(M)$ of the set $M = \{1, 2\}$, we introduce a relation R defined by the rule:

$$ARB \qquad \Longleftrightarrow \qquad A \cup \{1\} = B \cup \{2\}.$$

Write all ordered pairs in R.

7. On the set of real numbers $\mathbb{R}$, we define a relation $\sim$:

$$a \sim b \qquad \Longleftrightarrow \qquad a - b \in \mathbb{Z}.$$

Justify whether $\sim$ is an equivalence relation. If $\sim$ is an equivalence relation, find the equivalence classes $[\frac{1}{2}]_\sim$ and $[\sqrt{2}]_\sim$.

Mathematical Induction

8. Show that the number $3^{7^{999}} - 1$ is divisible by 2. Solve the problem using mathematical induction.

9. Prove that for every natural number $n > 1$, the equality

$$\left(1 - \frac{1}{4}\right) \cdot \left(1 - \frac{1}{9}\right) \cdot \left(1 - \frac{1}{16}\right) \cdot \ldots \cdot \left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}$$

holds.

10. Show that the sum $1 + 3 + 5 + 7 + \cdots + (2 \cdot 7^7 - 1)$ is divisible by 7^7. Solve the problem using mathematical induction.

11. Prove that the number $37^{500} - 37^{100}$ is divisible by 10.

All above math problems are taken from the following website:
`https://osebje.famnit.upr.si/~penjic/teaching.html`.
THE READER CAN FIND ALL SOLUTIONS TO THE GIVEN PROBLEMS ON THE SAME PAGE.